Matrix Decomposition Based Adaptive Control of Non-Canonical Form MIMO Discrete-Time Nonlinear Systems

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Abstract—This paper presents a new study on adaptive control of a class of non-canonical form MIMO discrete-time nonlinear systems with parametric uncertainties. To adaptively control such systems, this paper first reconstructs the non-canonical system dynamics with the output equation to derive a normal form by using a vector relative degree formulation. Then, a new matrix decomposition based adaptive control framework is developed for the controlled plant with vector relative degree \([1, 1, \ldots, 1]\) to achieve closed-loop stability and asymptotic output tracking under some relaxed design conditions, where the high-frequency gain matrix is handled by the matrix decomposition technique. An extension of the developed adaptive control framework to adaptive control of general canonical-form MIMO discrete-time nonlinear systems is also demonstrated. For the controlled plant with high-order vector relative degrees, this paper shows that the adaptive control problem faces the issues of nonlinear parametrization and non-affine control input, for which a new implicit function based nominal control scheme is proposed. Finally, an illustrative example is presented with simulation results to demonstrate the control system design procedure and the effectiveness of the proposed control scheme.

Index Terms—Adaptive control, implicit function, matrix decomposition, non-canonical form, output tracking

I. INTRODUCTION

For adaptive control of uncertain multi-input and multi-output (MIMO) nonlinear systems, it generally needs to deal with uncertain high frequency gain matrices. The adaptive versions of high frequency gain matrices may be singular in the process of parameter adaptation, which results in the singularity problem of adaptive control laws. To handle the singularity problem, the matrix decomposition technique was introduced in the year of 1993 by A. S. Morse in [1].

So far, a lot of remarkable results have been published addressing the singularity problem resulting from the high frequency gain matrices. For example, the matrix decomposition based adaptive control methods for linear time-invariant (LTI) systems covering continuous-time (CT) and discrete-time (DT) nonlinear systems have been well-developed, and a synthesis of this topic can be seen in [2]; the matrix decomposition based adaptive control for CT nonlinear systems have been deeply studied ([3]-[5]); and the matrix decomposition based adaptive fuzzy/neural network control methods for CT nonlinear systems have also been proposed ([8]-[12]). Specifically, the existing methods for LTI systems generally used the transfer function matrix for adaptive control design, the stability analysis is based on the small-gain lemma ([2]). For CT nonlinear systems, the existing methods mostly used the robust control technique and approximation (fuzzy/neural networks) technique. Note that the matrix decomposition based methods for CT LTI systems do not depend on the bound information of the high-frequency gains. However, those for CT nonlinear systems are not the case, and they generally need the bounds of the eigenvalues corresponding to the high frequency gains ([3]-[7]). Recently, without using the bound information, [8] and [12] gave valid solutions to matrix decomposition based adaptive control of MIMO CT nonlinear systems based on neural network and fuzzy based formulation, respectively.

However, it is rarely to see the application of the matrix decomposition technique to adaptive control of MIMO DT nonlinear systems, especially those in non-canonical forms. Non-canonical form means the system dynamics do not have strict-feedback forms, and the system output is generally a combination of some or all of the state variables. The existing methods for LTI systems cannot be extended to MIMO DT nonlinear systems as MIMO DT nonlinear systems do not have transfer functions, and there do not exist related small-gain lemmas that are applicable to MIMO DT nonlinear systems. Moreover, the existing methods of nonlinear systems mostly focus on canonical-form systems, which are not effective for control of non-canonical form case. The reason is the control methods for canonical-form nonlinear systems depend on the specific system structures, such as explicit relative degree or infinite zero structures (which are crucial for control designs), however, non-canonical form nonlinear systems do not have such features. Furthermore, the control methods proposed in [12] for non-canonical form CT system case also cannot be applicable to non-canonical form MIMO DT system case due to the essential differences between the system stability characterizations of CT and DT control schemes. On the other hand, the existing methods for adaptive control of MIMO DT nonlinear systems are mostly based on the triangular form inputs ([13]), which do not involve the singularity problem. Up to now, it is still open to address the singularity problem of
adaptable control law for MIMO DT nonlinear systems without triangular form inputs.

To sum up, how to develop a matrix decomposition based adaptive control scheme for non-canonical form MIMO DT nonlinear systems is still open for study. In this paper, we will systematically analyze this problem, and develop a matrix decomposition based adaptive control scheme for a class of non-canonical form MIMO DT nonlinear systems with vector relative degree \([1, 1, \ldots, 1]\) (the vector relative degree issue will be discussed in Section III, and the dimension of \([1, 1, \ldots, 1]\) is equal to the number of the system output variables).

For non-canonical form MIMO DT nonlinear systems with higher-order relative degrees, we will show that adaptive control of such systems further faces additional key technical issues. In 1989, S. S. Sastry and A. Isidori in [14] addressed a CT nonlinear system \(\dot{x} = f(x) + g(x)u, \ y = h(x)\), where \(x \in \mathbb{R}^n, \ u, y \in \mathbb{R}\) are the system state, input and output, respectively, and \(f(x) \in \mathbb{R}^n, \ g(x) \in \mathbb{R}^n, \ h(x) \in \mathbb{R}\) are nonlinear mappings with parametric uncertainties. The paper [14] developed an adaptive control scheme to ensure closed-loop stability and asymptotic output tracking for the nonlinear system. At the end of the paper, S. S. Sastry and A. Isidori clarified that the output tracking control of non-canonical form DT nonlinear systems is fairly complicated and open for study, which is also emphasized in [15]. Up to now, there are still no valid results to solve this problem. The main technical issues are (i) the output dynamics nonlinearly depend on the control signal, which leads to a difficulty in deriving an analytical adaptive control law; and (ii) the dependence of the output dynamics on unknown parameters is rather complex, which leads to a difficulty in estimating unknown parameters. Here, we will show that, when the non-canonical form MIMO DT nonlinear system considered in this paper has high-order vector relative degrees larger than \([1, 1, \ldots, 1]\), the adaptive control problem will face the above two technical issues. We will first discuss that the existing control methods are not effective to handle the problem, and then give a nominal control solution to this problem. In summary, the main contributions of this paper are as follows.

(i) This paper is the first to establish a linearly parametrized adaptive control framework to deal with the control law singularity problem in adaptive control of a class of non-canonical form MIMO DT nonlinear systems.

(ii) A matrix decomposition based adaptive state feedback control scheme is developed for a class of MIMO DT nonlinear systems with vector relative degree \([1, 1, \ldots, 1]\), which solves the adaptive control law singularity problem and ensures closed-loop stability and asymptotic output tracking. We also show that the proposed scheme is applicable to a general class of uncertain canonical-form nonlinear systems with a general vector relative degree.

(iii) A nominal state feedback control scheme is proposed for a class of non-canonical form MIMO DT nonlinear systems with high-order vector relative degrees, as a fundamental study for adaptive control design.

The rest of this paper is organized as follows. Section II presents the system model and formulates the problems to be solved; Section III addresses the feedback linearization of the controlled plant; Section IV gives the details of the matrix decomposition based adaptive state feedback control scheme, and also discusses an extension of the proposed control scheme. Section V presents the details of the nominal control design for systems with high-order vector relative degrees; Section VI is the simulation part; and finally Section VII gives the concluding remarks.

II. Problem Statement

This section presents the system model and the problems to be solved in this paper.

A. System Model

Consider the following multi-input and multi-output (MIMO) DT nonlinear system:

\[
x(t + 1) = \Theta_f(x(t)) + Bu(t), \hspace{1cm} y(t) = Cx(t),
\]

where \(t \in \{0, 1, 2, \ldots, \}, \ x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n, \ y(t) = [y_1(t), y_2(t), \ldots, y_M(t)]^T \in \mathbb{R}^M, \ u(t) = [u_1(t), u_2(t), \ldots, u_M(t)]^T \in \mathbb{R}^M\) are the state vector, output vector, and input vector, respectively, \(B = [B_1, B_2, \ldots, B_M] \in \mathbb{R}^{n \times M}\) with \(B_j = [b_{j1}, b_{j2}, \ldots, b_{jn}]^T \in \mathbb{R}^n\), \(C = [C_1^T, C_2^T, \ldots, C_M^T] \in \mathbb{R}^{M \times n}\) with \(C_j = [c_{j1}, c_{j2}, \ldots, c_{jn}]^T \in \mathbb{R}^n\) and

\[
\Theta_f = \begin{bmatrix}
\Theta_{f_1}^T \\
\Theta_{f_2}^T \\
\vdots \\
\Theta_{f_n}^T
\end{bmatrix} \in \mathbb{R}^{n \times \sum_{i=1}^n p_i},
\]

\[
\phi_f(x(t)) = [\phi_{f_1}^T(x(t)), \phi_{f_2}^T(x(t)), \ldots, \phi_{f_n}^T(x(t))]^T \in \mathbb{R}^{\sum_{i=1}^n p_i}
\]

for \(\Theta_{f_i} = [\theta_{f_{i1}}, \theta_{f_{i2}}, \ldots, \theta_{f_{ip_i}}]^T \in \mathbb{R}^{p_i}\) and

\[
\phi_f(x(t)) = [f_{11}(x(t)), f_{12}(x(t)), \ldots, f_{ip_i}(x(t))]^T \in \mathbb{R}^{p_i}.
\]

Note that all other elements (that are not given) of \(\Theta_f^*\) are zero. In this paper, we assume that \(B, C\) are unknown constant matrices, \(\theta_{f_i}^*, i = 1, 2, \ldots, n\), \(k = 1, 2, \ldots, p_i\), are unknown constant parameters, and \(f_{ik} : \mathbb{R}^n \rightarrow \mathbb{R}\) are known mappings such that \(f_{ik}(X)\) are globally Lipschitz in \(X\) with \(X\) denoting any signal belonging to \(\mathbb{R}^n\).

From the system structure, we see that the system (1) is in a non-canonical form with linearly parametrized uncertainties. In this paper, we assume that the system state is measurable.

Remark 1: The system (1) can describe a wide class of nonlinear control systems with parametric uncertainties. Actually, for a nonlinear system \(x(t + 1) = f(x(t)) + \sum_{i=1}^N B_i u_j(t), \ y_j(t) = C_j x(t), j = 1, 2, \ldots, M\), as long as \(f(x(t))\) only contains linearly parametrized uncertainties, one can always rewrite this system to the form (1). Moreover, the results derived in this paper are applicable to a general class of non-canonical form DT nonlinear systems of the form

\[
x(t + 1) = f(x(t)) + g(x(t))u(t), \hspace{1cm} y_j(t) = C_j x(t), k = 1, 2, \ldots, M,
\]
where \( g(x(t)) = [g_1(x(t)), g_2(x(t)), \ldots, g_n(x(t))]^T \in \mathbb{R}^{n \times M} \) such that \( g_i : \mathbb{R}^n \rightarrow \mathbb{R}^M \) are continuously differentiable mappings with linearly parametrized uncertainties. The control design for the system (5) will involve too much notation to follow, thus, to simplify the notation used in this paper, we use the model (1) for adaptive control design, and explain how to extend the control method for the system (1) to the general system (5) where necessary.

**B. Research Problems**

**Control objective.** The control objective of this paper is to develop an adaptive state feedback control scheme for the system (1) to ensure closed-loop stability and asymptotic output tracking under some mild design conditions.

To meet the control objective, the followings show the technical issues to be solved. Note that the system (1) is in a non-canonical form and not suitable for adaptive control design, so we first need to re-construct the system dynamics in prior. In this paper, we plan to use a relative degree based reconstruction method to solve this problem. The relative degree concept has been studied in [24] for SISO DT nonlinear systems. In this paper, we will extend the relative degree concept to the MIMO DT nonlinear system (1), based on which the system (1) will be transformed into two subsystems: output dynamics and internal dynamics (the details will be shown in Section III). However, the transformed system will face new technical issues. Here, we illustrate two representative relative degree cases to show the characterizations of the adaptive control problem addressed in this paper.

**Specification of the control problem.** If the system (1) has vector relative degree \([1, 1, \ldots, 1]\) for all \((x, u) \in \mathbb{R}^n \times \mathbb{R}^M\), then the output dynamics are

\[
y(t + 1) = \Theta_{\epsilon_f}^* \phi_f(x(t)) + \Theta_{\epsilon_b}^* u(t),
\]

where \( \Theta_{\epsilon_b}^* \) is nonsingular, and \( \Theta_{\epsilon_f}^* \) and \( \Theta_{\epsilon_b}^* \) are unknown of the forms

\[
\Theta_{\epsilon_f}^* = [\theta_{\epsilon_1 f}, \theta_{\epsilon_2 f}, \ldots, \theta_{\epsilon_M f}]^T \in \mathbb{R}^{M \times \sum_{i=1}^n p_i}, \quad \Theta_{\epsilon_b}^* = [\theta_{\epsilon_1 b}^*, \theta_{\epsilon_2 b}^*, \ldots, \theta_{\epsilon_M b}^*]^T \in \mathbb{R}^{M \times M}, \quad \theta_{\epsilon_{ij}}^* = C_j \Theta_{\epsilon_f}^* = [c_{j1} \theta_{\epsilon_{f1}}^T, c_{j2} \theta_{\epsilon_{f2}}^T, \ldots, c_{jM} \theta_{\epsilon_{fM}}^T]^T \in \mathbb{R}^{\sum_{i=1}^n p_i}, \quad \theta_{\epsilon_{ij}}^* = [C_j B_1, C_j B_2, \ldots, C_j B_M]^T \in \mathbb{R}^{M}.
\]

A conventional adaptive control law for (6) can be designed as

\[
u(t) = \Theta_{\epsilon_b}^{-1} (t) (-\Theta_{\epsilon_f}^{-1} (t) \phi_f (x(t)) + v(t)),
\]

where \( \Theta_{\epsilon_f}^{-1} (t) \) and \( \Theta_{\epsilon_b}^{-1} (t) \) are estimates of \( \Theta_{\epsilon_f}^* \) and \( \Theta_{\epsilon_b}^* \), respectively, and \( v(t) \) is a designed signal. Since the high frequency gain \( \Theta_{\epsilon_b}^* \) is a matrix, for adaptive control design, the estimate \( \Theta_{\epsilon_b}^{-1} (t) \) may be singular in the process of parameter adaptation, which results in the singularity of the adaptive control law. As clarified in Introduction, up to now, to develop a matrix decomposition based adaptive control method under a linearly parametric framework for MIMO DT nonlinear systems is open. This is the first main technical issue to be solved.

If the system (1) has any higher-order vector relative degrees, the situation will be essentially different. As an example, if the system (1) has vector relative degree \([2, 2, \ldots, 2]\) for all \((x, u) \in \mathbb{R}^n \times \mathbb{R}^M\), then \( \Theta_{\epsilon_b}^* = 0 \) and the output dynamics are

\[
y(t + 1) = \Theta_{\epsilon_f}^* \phi_f(x(t)),
\]

\[
y(t + 2) = \Theta_{\epsilon_f}^* \phi_f(x(t + 1)) = \Theta_{\epsilon_f}^* \phi_f(\Theta_{\epsilon_f}^* \phi_f(x(t)) + Bu(t)),
\]

such that \( \Theta_{\epsilon_f}^* \partial \phi_f(\Theta_{\epsilon_f}^* \phi_f(x(t)) + Bu(t)) \) is nonsingular for all \((x, u) \in \mathbb{R}^n \times \mathbb{R}^M\). Note that \( \phi_f \) is a nonlinear mapping, which follows from (12) that \( y(t + 2) \) linearly depends on \( \Theta_{\epsilon_f}^* \), but nonlinearly depends on \( \Theta_{\epsilon_f}^* \), \( B \) and \( u(t) \). Such characterizations bring two difficulties: one is how to simultaneously handle the linearly and nonlinearly parametrized uncertainties in \( y(t + 2) \); and the other is how to derive an explicit adaptive control law. This is also the reason why S. S. Sastry and A. Isidori said ([14]) that adaptive control of non-canonical form DT nonlinear systems is fairly complicated.

**Existing control methods cannot solve the control problem in this paper.** As clarified in Introduction, the existing methods addressing the singularity problem of MIMO CT nonlinear systems are not effective for control of the MIMO DT system (1) with vector relative degree \([1, 1, \ldots, 1]\). Particularly, the singularity problem in adaptive control of MIMO DT nonlinear systems has not been studied before in the literature.

On the other hand, there exists a vast number of literature addressing adaptive control for systems with nonlinearly parametrized uncertainties or non-affine control inputs ([16]-[21]). In the literature, there are two main methods to handle the nonlinearly parametrized uncertainties: (i) one is to assume that the functions containing nonlinearly parametrized uncertainties (denoted as \( \theta^* \)) should satisfy some design conditions, such as Lyapunov based condition, etc., with respect to \( \theta^* \), so that \( \theta^* \) could satisfy some inequalities which are suitable for adaptive control designs ([16]); and (ii) the other is to use some approximation techniques, such as neural network or fuzzy systems, to approximately parametrize the functions containing the nonlinearly parametrized uncertainties ([17]-[19]). While, the main method to handle the non-affine control input is to use some approximation techniques, such as neural network or fuzzy systems, to approximately express the control input of a non-affine form into a linear form in the system dynamics ([21]).

However, the nonlinearly parametrized uncertainties handled in this paper have analytical expressions and the high-order output dynamics also have the intrinsic structure, as shown in (12). Based on the analytical expressions of the nonlinearly parametrized uncertainties in the output dynamics, if we do not use approximation or make additional conditions on the output dynamics, one can verify that the unknown parameters in the output dynamics do not satisfy those design conditions specified in the literature. In other words, adaptive control of non-canonical form MIMO DT nonlinear systems with high-order vector relative degrees is also open.

**Technical issues to be solved in this paper.** In this paper, we mainly address the matrix decomposition based adaptive control problem for the system (1) with vector relative degree \([1, 1, \ldots, 1]\). For the high-order vector relative degree case, we
present a fundamental study by establishing a nominal control framework for the system (1) with a high-order vector relative degree. Thus, the following technical issues need to be solved:

- how to re-construct the non-canonical system dynamics so as to be suitable for adaptive control designs;
- how to develop a matrix decomposition based adaptive control law for the system (1) with vector relative degree \([1,1,\ldots,1]\) to ensure desired system performance; and
- how to design a matrix decomposition based nominal control scheme for the system (1) with a general high-order vector relative degree larger than \([1,1,\ldots,1]\).

III. FEEDBACK LINEARIZATION OF DT NONLINEAR SYSTEMS

This section addresses feedback linearization of the system (1), by specifying the general vector relative degree and deriving a vector relative degree dependent normal form. Then, a design condition on the normal form is proposed, which is crucial for stability analysis.

A. Vector Relative Degrees of MIMO DT Nonlinear Systems

Define

\[
F(x,u) = \Theta_j \phi_j(x(t)) + Bu(t), \quad F_0(x) = F(x,0).
\] (13)

Introduce a notation \(\circ\) to denote a composition operation, that is, \(p_1 \circ p_2\) denotes that \(p_1\) is a function of \(p_2\) for any functions \(p_1\) and \(p_2\) of appropriate dimensions. Specifically, for any positive integer \(k\), \(F_0^k \circ F(x,u) = F_0(F_0^{k-1} \circ F(x,u))\) with \(F_0^0 \circ F(x,u) = F(x,u)\) and \(F_0 \circ F(x,u) = F(F(x,u),0)\).

Define a matrix

\[
G = \begin{bmatrix}
C_1 \frac{\partial F_0^{\rho_1-1}}{\partial u} \circ F(x,u) \\
C_2 \frac{\partial F_0^{\rho_2-1}}{\partial u} \circ F(x,u) \\
\vdots \\
C_M \frac{\partial F_0^{\rho_M-1}}{\partial u} \circ F(x,u)
\end{bmatrix} \in \mathbb{R}^{M \times M}
\] (14)

which will be used for defining vector relative degrees.

Motivated by the definitions of SISO DT nonlinear systems proposed in [24] and MIMO CT nonlinear systems proposed in [25], we make the following definition.

**Definition 1:** The system (1) has vector relative degree \([\rho_1, \rho_2, \ldots, \rho_M]\) (\(\rho_j \geq 1\), \(\sum_{j=1}^{M} \rho_j \leq n\)) for all \((x,u) \in \mathbb{R}^n \times \mathbb{R}^M\), if

\[
C_j \frac{\partial F_0^{k_j}}{\partial u} \circ F(x,u) = 0, \quad k_j = 0, 1, \ldots, \rho_j - 2, \quad j = 1, 2, \ldots, M;
\] (15)

and \(G\) in (14) is nonsingular for all \((x,u) \in \mathbb{R}^n \times \mathbb{R}^M\).

This definition specifies a general vector relative degree condition for the system (1), and can be easily extended to define relative degrees of a general class of MIMO nonlinear systems of the form (5), which just involves more expressions.

B. Normal Form of DT Nonlinear Systems

Now, we show that the non-canonical system dynamics with the output equation can be transformed into two subsystems, assuming the system (1) has vector relative degree \([\rho_1, \rho_2, \ldots, \rho_M]\).

**Lemma 1:** If the system (1) has vector relative degree \([\rho_1, \rho_2, \ldots, \rho_M]\) for all \((x,u) \in \mathbb{R}^n \times \mathbb{R}^M\), via a diffeomorphism \(T(x(t)) = [\xi^T(t), \eta^T(t)]^T\) for \(\xi(t) = [\xi_1^T(t), \xi_2^T(t), \ldots, \xi_M^T(t)]^T \in \mathbb{R}^{\sum_{i=1}^{M} \rho_i}\) with \(\xi(0) = [\xi_{j1}, \xi_{j2}, \ldots, \xi_{jp_j}]^T \in \mathbb{R}^{\rho_j}\), and \(\eta(t) \in \mathbb{R}^{n-\sum_{j=1}^{M} \rho_j}\), then the system (1) can be transformed into two subsystems: the output dynamics

\[
\xi_j(t+1) = \xi_{j+1}(t), \quad i = 1, \ldots, \rho_j - 1,
\]

\[
\xi_{jp_j}(t+1) = C_j F_0^{\rho_j-1} \circ F(x(t),u(t))
\] (16)

with \(\xi_j(t+1) = y_j(t), \quad j = 1, 2, \ldots, M\), such that \(G\) in (14) is non-singular for all \((x,u) \in \mathbb{R}^n \times \mathbb{R}^M\); and the internal dynamics

\[
\eta(t+1) = q(\xi(t), \eta(t), u(t)),
\] (17)

where \(q : \mathbb{R}^{\sum_{j=1}^{M} \rho_j} \times \mathbb{R}^{n-\sum_{j=1}^{M} \rho_j} \times \mathbb{R}^M \rightarrow \mathbb{R}^{n-\sum_{j=1}^{M} \rho_j}\) is a nonlinear mapping.

To prove Lemma 1, we need to specify \(\xi(t)\) and \(\eta(t)\) to satisfy (16) and (17). Since \(\xi_j(t) = y_j(t)\), it is easy to specify \(\xi(t)\) as \(\xi(t) = [\xi_{11}, \xi_{12}, \cdots, \xi_{M1}, \xi_{M2}, \cdots, \xi_{M\rho_M}]^T\). Based on Frobenius Theorem (see Theorem 1.4.1 in [25]), one can find \(n-\sum_{j=1}^{M} \rho_j\) vectors to construct \(\eta(t)\). The proof of Lemma 1 is similar to that of CT nonlinear systems case. One may refer to [25], and we omit the proof for simplicity. Lemma 1 provides a vector relative degree dependent normal form which will be used for adaptive output tracking control design.

C. Input-to-State Stable Condition on Internal Dynamics

For output tracking control design, the control law \(u\) is designed of the basic form \(u(t) = u(x(t), y^*(t)) = u(T^{-1}(\xi(t), \eta(t)), y^*(t))\), where \(y^*(t)\) is a given bounded reference output signal. Thus, (17) can be expressed as

\[
\eta(t+1) = q(\xi(t), \eta(t), y^*(t)).
\] (18)

Since \(y^*(t)\) is bounded, it can be seen as an external input signal for the system (18). We make the following assumption.

**Assumption 1:** The origin of the system \(\eta(t+1) = q(0, \eta(t), 0)\) is globally exponentially stable, and \(q(\xi, \eta, v)\) is globally Lipschitz in \(\xi\) and \(v\).

**Remark 2:** Assumption 1 is often called the input-to-state stable (ISS) condition or strong minimum phase condition in the references ([22], [23]). Based on Assumption 1,

\[
\|q(\xi(t), \eta(t), v(t)) - q(0, \eta(t), 0)\| \\
+ \|q(0, \eta(t), v(t)) - q(0, \eta(t), 0)\| \\
\leq L_\xi \|\xi(t)\| + L_v \|v(t)\| + L_0,
\] (19)

where \(L_\xi\) and \(L_v\) are Lipschitz constants, \(L_0\) is some positive constant, and \(\|X\|\) denotes the Euclidean norm of \(X\) of any appropriate dimension. The inequality (19) implies that the system (18) is input-to-state stable, that is, if \(\xi(t)\) and \(v(t)\) are bounded, \(\eta(t)\) is bounded. For stable output tracking control design of nonlinear systems covering CT and DT, the ISS design condition is essential even for the case of systems without
unknown parameters. In this paper, for adaptive control design of non-canonical form DT systems, the ISS design condition is also needed. Note that many applications are with ISS internal dynamics, such as a high fidelity transport mode [26]. The ISS design condition is an extension of a fundamental design condition for model reference adaptive control (MRAC) of LTI systems: the zeros of the transfer function are stable. □

IV. ADAPTIVE CONTROL DESIGN FOR SYSTEMS WITH $\rho_1 = 1$

This section first presents an adaptive control scheme for the system (1) with vector relative degree $[1,1,...,1]$, and then, gives an extension of the adaptive control scheme to canonical-form MIMO DT nonlinear systems with a general vector relative degree.

When the system (1) has vector relative degree $[1,1,...,1]$, from (6), the output dynamics is

$$y(t + 1) = \Theta_{c, f}^*\phi_f(x(t)) + \Theta_{cb}^*u(t).$$

Before proceeding the control design, we first make an assumption which was used for decomposition of the high frequency gain matrix $\Theta_{cb}^*$.

**Assumption 2:** All leading principal minors of $\Theta_{cb}^*$, defined as $\Delta_i$, $i = 1,2,...,M$, are nonzero and their signs are known, that is, $\Delta_i \neq 0$, sign($\Delta_i$) are known.

**Remark 3:** Note that there is no need to be uniform of the signs of $\Delta_i$. Assumption 2 can be seen as an extension of a fundamental design condition for matrix decomposition based MRAC of LTI systems: all leading principal minors of the high frequency gain matrix are nonzero and their signs are known (see Assumption 9.3 in [21]). Based on Assumption 2, $\Theta_{cb}^*$ can be uniquely decomposed as $\Theta_{cb}^* = LD^*U^*$, for some lower triangular matrix $L$, some upper triangular matrix $U$, and

$$D^* = \text{diag}\{d_1^*, d_2^*, ..., d_M^*\} = \text{diag}\{\frac{\Delta_2}{\Delta_1}, ..., \frac{\Delta_M}{\Delta_{M-1}}\}. \quad (21)$$

Furthermore, we can derive the SDU decomposition as $\Theta_{cb} = S^*D_*U_s$, where $S^* = LD^*D_*^{-1}L^T$ is a positive definite matrix, $D_* = D_*^{-1}L^{-1}T$ $D_*U$ is still a unit upper triangular matrix, and

$$D_* = \text{diag}\{s_1^*, s_2^*, ..., s_M^*\} = \text{diag}\{\text{sign}\{d_1^*\gamma_1, \text{sign}\{d_2^*\gamma_2, ..., \text{sign}\{d_M^*\gamma_M\}}\} \quad (22)$$

such that $\gamma_i > 0$, $i = 1, ..., M$, may be arbitrary. □

**Remark 4:** In Assumption 2, we assume the signs of $\Delta_i$ to be known. Actually, there exist some methods that can relax the control gain sign condition, such as Nussbaum and multiple-model techniques ([30], [31]). In this paper, we do not consider the unknown control gain sign problem which needs further study. □

**Remark 5:** For the general non-canonical form nonlinear system (5) with vector relative degree $[1,1,...,1]$, the output dynamics can be expressed as $y(t + 1) = \Theta_{c, f}^*\phi_f(x(t)) + \Theta_{cb}^*\phi_y(x(t))u(t)$, where $\Theta_{c, f}^*$, $\Theta_{cb}^*$ are unknown and $\phi_f(x(t))$, $\phi_y(x(t))$ are known of appropriate dimensions. The corresponding assumption for matrix decomposition is that $\Theta_{cb}^*$ is a square matrix, and all leading principal minors of $\Theta_{cb}^*$ are nonzero and their signs are known. Then, the adaptive control design for the system (5) can be quite similar to that for the system (1). □

A. Parametrized Model

For the system (1) with vector relative degree $[1,1,...,1]$ for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^M$, from Lemma 1 and (6), we have

$$y(t + 1) = \Theta_{c, f}^*\phi_f(x(t)) + \Theta_{cb}^*u(t), \quad (23)$$

where $\Theta_{cb}^*$ is nonsingular. Based on Assumption 2 and Remark 3, (23) can be expressed as

$$S^*^{-1}y(t + 1) = S^*^{-1}\Theta_{c, f}^*\phi_f(x(t)) + D_*U_su(t), \quad (24)$$

where $S^*$ is a positive definite matrix, $U_s$ is a unit upper triangular matrix, and $D_*$ is a known diagonal matrix of the form $D_* = \text{diag}\{\text{sign}\{d_1^*\gamma_1, \text{sign}\{d_2^*\gamma_2, ..., \text{sign}\{d_M^*\gamma_M\}\}}\}$ such that $\gamma_i > 0$, $i = 1, ..., M$, may be arbitrary. From (24), we derive the parametrized model as

$$S^*^{-1}y(t + 1) = D_*\Theta_1^*\phi_f(x(t)) + D_*\Theta_2^*u(t) + D_*u(t), \quad (25)$$

where $\Theta_1^* = D_*^{-1}S^*^{-1}\Theta_{c, f}^*$, and $\Theta_2^* = U_s - I$ of the form

$$\Theta_2^* = \begin{bmatrix} 0 & \theta_{212} & \cdots & \theta_{21M} \\ 0 & 0 & \cdots & \theta_{22M} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (26)$$

For $\Theta_2^*$ in (26), we define the following vectors

$$\theta_{2j} = [\theta_{2j1}, \theta_{2j2}, ..., \theta_{2jM}]^T \in \mathbb{R}^{M-1}, \quad j = 1, ..., M - 1. \quad (27)$$

**Remark 6:** Note that we divide $U_s$ into two parts: $I$ and $U_s - I$. It aims to solve the possible singularity of adaptive version of $U_s$. We will show that, based on this manipulation, the control law can be designed as a particular structure which ensures a well-defined control law. □

B. Adaptive Control Law

To ensure desired system performance, the adaptive control law is designed as

$$u(t) = -\Theta_2(t)u(t) - \Theta_1(t)\phi_f(x(t)) + \Theta_3(t)y^*(t + 1) - \Theta_3(t)A_m(y(t) - y^*(t)), \quad (28)$$

where $\Theta_i(t), i = 1,2,3$, are the estimates of $\Theta_1^*$, $\Theta_2^*$, $(S^*D_*)^{-1}$, respectively, $A_m$ is a chosen stable matrix, and $y^*(t) \in \mathbb{R}^M$ is a given bounded reference output signal. In particular, $\Theta_3(t)$ has the form

$$\Theta_3(t) = \begin{bmatrix} 0 & \theta_{212}(t) & \cdots & \theta_{21M}(t) \\ 0 & 0 & \cdots & \theta_{22M}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (29)$$

where $\theta_{2j}(t)$ are estimates of $\theta_{2j}$, respectively. The control signal can be calculated as follows. Letting $v(t) =$
\[ [v_1(t), v_2(t), \ldots, v_M(t)]^T = -\Theta_1(t)\phi_f(x(t)) + \Theta_3(t)y^*(t + 1) - D_s A_m(y(t) - y^*(t)), \]
we have
\[ u_M(t) = v_M(t), \]
\[ u_{M-1}(t) = -\theta_{2(M-1),t} u_M(t) + v_{M-1}(t), \]
\[ \vdots \]
\[ u_1(t) = -\theta_{211(t)} u_2(t) - \cdots - \theta_{21M(t)} u_M(t) + v_1(t). \]
With the special form of \( \Theta_i(t) \) in (29), the adaptive control law (28) is implementable without the singularity problem.

C. Tracking Error Model

Define the tracking error
\[ e(t) = y(t) - y^*(t) \in \mathbb{R}^M. \]
Substituting (28) to (25) gives the following equation
\[ S^{*\dagger} y(t + 1) = -D_s \bar{\Theta}_1(t) \phi_f(x(t)) - D_s \bar{\Theta}_2(t) u(t) + D_s \Theta_3(t) y^*(t + 1) - D_s \Theta_3(t) A_m e(t), \]
where \( \bar{\Theta}_i(t) = \Theta_i(t) - \Theta_i^* \), \( i = 1, 2 \). Adding \(-S^{*\dagger} y^*(t + 1) + S^{*\dagger} A_m e(t) \) to both sides of (32), with some further manipulations, we have
\[ S^{*\dagger} (e(t + 1) + A_m e(t)) = -D_s (\bar{\Theta}_1(t) \phi_f(x(t)) + \bar{\Theta}_2(t) u(t) + \Theta_3(t) A_m e(t) - y^*(t + 1))), \]
where \( \bar{\Theta}_3(t) = \Theta_3(t) - \Theta_3^* \). Define
\[ \Psi(t) = [\Theta_1(t), \Theta_2(t), \Theta_3(t)], \]
\[ \bar{\Psi}(t) = \Psi(t) - \Psi^*(t) = [\bar{\Theta}_1(t), \bar{\Theta}_2(t), \bar{\Theta}_3(t)], \]
\[ \varphi(t) = [-\bar{\phi}_f^T(x(t)), -u^T(t), -(A_m e(t) - y^*(t + 1))^T]^T. \]
\[ P_m(z) = z I + A_m, \]
where \( z \) denotes the time advance operator, that is,
\[ z[X](t) = X(t + 1), \]
\[ \sum_{i=0}^{p} a_i z^i[X](t) = \sum_{i=0}^{p} a_i X(t + i), \]
where \( a_i \) are constants. Then, (33) can be expressed as
\[ P_m(z)[e](t) = S^* D_s \bar{\Psi}(t) \varphi(t). \]
To implement \( u \) from (30), we express \( \bar{\Psi}(t) \phi(t) \) as
\[ \bar{\Psi}(t) \phi(t) = \begin{bmatrix} (\Psi_1(t) - \Psi_1^*) \varphi_1(t) \\ (\Psi_2(t) - \Psi_2^*) \varphi_2(t) \\ \vdots \\ (\Psi_M(t) - \Psi_M^*) \varphi_M(t) \end{bmatrix}, \]
where
\[ \Psi_1(t) - \Psi_1^* = [\theta_{212(t)} - \theta_{212}^*, \ldots, \theta_{21M(t)} - \theta_{21M}^*], \]
\[ \bar{\Theta}_1(t), \bar{\Theta}_1(t), \]
\[ \Psi_2(t) - \Psi_2^* = [\theta_{222(t)} - \theta_{222}^*, \ldots, \theta_{22M(t)} - \theta_{22M}^*], \]
\[ \bar{\Theta}_2(t), \bar{\Theta}_2(t), \]
\[ \vdots \]
\[ \Psi_{M-1}(t) - \Psi_{M-1}^* = [\theta_{2(M-1),t} - \theta_{2(M-1),t}^*], \]
\[ \bar{\Theta}_{M-1}(t), \bar{\Theta}_{M-1}(t), \]
\[ \Psi_M(t) - \Psi_M^* = [\bar{\Theta}_M(t), \bar{\Theta}_M(t)], \]
\[ \bar{\Theta}_M(t), \bar{\Theta}_M(t), \]
and
\[ \varphi_1(t) = [-u_2(t), -u_3(t), \ldots, -u_M(t), -\phi_f^T(x(t)), -A_m e(t) - y^*(t + 1))^T]^T, \]
\[ \varphi_2(t) = [-u_2(t), \ldots, -u_M(t), -\phi_f^T(x(t)), -A_m e(t) - y^*(t + 1))^T]^T, \]
\[ \vdots \]
\[ \varphi_{M-1}(t) = [-u_2(t), \ldots, -u_M(t), -\phi_f^T(x(t)), -A_m e(t) - y^*(t + 1))^T]^T, \]
\[ \varphi_M(t) = [-\phi_f^T(x(t)), -(A_m e(t) - y^*(t + 1))^T]^T. \]
Introduce a stable polynomial
\[ h_0(z) = z - \alpha \]
with \( 0 < \alpha < 1 \). Then, define
\[ h(z) = \frac{1}{h_0(z)}. \]
To derive the parameter update laws, we need a filtered error defined as
\[ \bar{e}(t) = P_m(z) h(z) e(t) = [\bar{e}_1(t), \ldots, \bar{e}_M(t)]^T. \]
Letting \( h(z) \) operate both sides of (40), we obtain
\[ \bar{e}(t) = S^* D_s h(z) [\bar{\Psi} \varphi(t)] \]
which is the expected tracking error model.

D. Estimation error

To design the update laws, we define an estimation error:
\[ \epsilon(t) = \bar{e}(t) + \Phi(t) \sigma(t), \]
where \( \sigma(t) = [\sigma_1(t), \sigma_2(t), \ldots, \sigma_M(t)]^T \), \( \Phi(t) \) is the estimate of \( S^* D_s \), and, for \( j = 1, 2, \ldots, M \),
\[ \sigma_j(t) = \Psi_j(t) \delta_j(t) - h(z) [\Psi_j \varphi_j(t)], \]
\[ \delta_j(t) = h(z) [\varphi_j(t)]. \]
Note that \( \bar{e}(t), \epsilon(t), \sigma_j(t), \delta_j(t) \) are all available at the current time instance. From (53)-(56), \( \epsilon(t) \) can be expressed as
\[ \epsilon(t) = S^* D_s [\bar{\Psi}_1 \delta_1(t), \ldots, \bar{\Psi}_M \delta_M(t)]^T + \bar{\Phi}(t) \sigma(t) \]
which will be used for stability analysis.

E. Parameter Update Laws

To implement the control signal, from (28), we only need to update \( \Theta_i \), \( i = 1, 2, 3 \). It follows from (34) that updating \( \Theta_i \), \( i = 1, 2, 3 \), is equivalent to updating \( \Psi(t) \). In addition to \( \Phi(t) \) the estimate of \( S^* D_s \), we need to design the parameter update laws to update \( \Psi(t) \) and \( \Phi(t) \). Using (54)-(56), the
parameter update laws are designed as
\[ \Psi_j^T(t+1) = \Psi_j^T(t) - \frac{\text{sign}[d_j^*]}{m_T(t)} \lambda \gamma_i \delta_j(t), \quad j = 1, 2, \ldots, M, \]  
(58)
\[ \Phi(t+1) = \Phi(t) - \frac{\beta}{m^2(t)} \sigma(t), \]
(59)
where
\[ m(t) = \sqrt{1 + \sum_{j=1}^M \sigma_j^2(t) + \sum_{j=1}^M \delta_j^2(t) \delta_j(t)}, \]
(60)
and, for \( j = 1, 2, \ldots, M \), sign\([d_j^*]\) are specified in (21) and (22), \( \gamma \) are the chosen parameters used in (22) such that \( 0 < \gamma_j^2 < 2\lambda_{\min}\{S^{-1}\} \), and \( \beta \in \mathbb{R} \) is a chosen adaptive gain such that \( 0 < \beta < \frac{2\lambda_{\min}\{S^{-1}\}}{\lambda_{\max}\{S^{-1}\}} \) with \( \lambda_{\min}\{S^{-1}\} \) and \( \lambda_{\max}\{S^{-1}\} \) denoting the minimum and maximum eigenvalues of \( S^{-1} \).

**Remark 7:** Note that \( \gamma_j, \ j = 1, 2, \ldots, M \), are chosen constant parameters, and
\[ S^{-1} = L^{-1}T^T D_T D_T^{-1} L^{-1} \]
\[ = L^{-1}T^T \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_M \end{bmatrix} \begin{bmatrix} |d_1| & |d_2^r| & \cdots & |d_M| \end{bmatrix} L^{-1}. \]
(61)
Then, \( 0 < \gamma_j^2 < 2\lambda_{\min}\{S^{-1}\} \) is satisfied when \( \gamma_j \in (0, \min\{|d_1|^2, \cdots, |d_M|^2\}) \). Moreover, \( \beta \) is also a chosen constant parameter, so it is feasible to specify \( \beta \) such that \( 0 < \beta < \frac{2\lambda_{\min}\{S^{-1}\}}{\lambda_{\max}\{S^{-1}\}} \). Thus, the design conditions on \( \gamma_j \) and \( \beta \) in the parameter update laws can be met.

Now, we derive the following result.

**Lemma 2:** The parameter update laws (58) and (59) ensure, for \( j = 1, 2, \ldots, M \),

\( i \) \( \Psi_j(t) \in L^\infty, \Phi(t) \in L^\infty \);

\( ii \) \( \frac{\epsilon(t)}{m(t)} \in L^2 \cap L^\infty, \Psi_j(t+1) - \Psi_j(t) \in L^2 \cap L^\infty, \Phi(t+1) - \Phi(t) \in L^2 \cap L^\infty, \) and

\( iii \) \( \lim_{t \to \infty} \frac{\epsilon(t)}{m(t)} = 0, \lim_{t \to \infty} (\Psi_j(t+1) - \Psi_j(t)) = 0, \lim_{t \to \infty} (\Phi(t+1) - \Phi(t)) = 0. \)

**Proof:** Consider a positive definite function
\[ V(\tilde{\Psi}_j(t), \tilde{\Phi}(t)) = \sum_{j=1}^M \tilde{\Psi}_j(t) \tilde{\Psi}_j^T(t) + \frac{1}{\beta} \text{tr}[\tilde{\Phi}_j^T(t) S^{-1}(t) \Psi_j(t)]. \]
(62)
Then,
\[ V(\tilde{\Psi}_j(t+1), \tilde{\Phi}(t+1)) - V(\tilde{\Psi}_j(t), \tilde{\Phi}(t)) \]
\[ = \sum_{j=1}^M \tilde{\Psi}_j(t+1) \tilde{\Psi}_j^T(t+1) - \sum_{j=1}^M \tilde{\Psi}_j(t) \tilde{\Psi}_j^T(t) \]
\[ + \frac{1}{\beta} \text{tr}[\tilde{\Phi}_j^T(t+1) S^{-1}(t+1) \Phi(t+1)] - \frac{1}{\beta} \text{tr}[\tilde{\Phi}_j^T(t) S^{-1}(t) \Phi(t)] \]
\[ = -\sum_{j=1}^M \left( \frac{2\text{sign}[d_j^*] \lambda_{\gamma_i} \delta_j(t) \delta_j(t)}{m(t)} - \frac{\gamma_j^2}{m^2(t)} \delta_j(t) \delta_j(t) \right) \]
\[ - \text{tr} \left[ \frac{2\Phi(t) S^{-1}(t) \sigma(t)}{m^2(t)} - \beta \text{tr}[\sigma(t) \sigma(t)] S^{-1}(t) \Phi(t) \right]. \]
(63)
Using the properties of matrix trace: \( \text{tr}[X_1] = \text{tr}[X_1^T] \) and \( \text{tr}[X_2 X_3] = \text{tr}[X_2^T X_3] \) of any matrices \( X_1, X_2, X_3 \), of appropriate dimensions, we derive
\[ \text{tr} \left[ \frac{2\Phi(t) S^{-1}(t) \sigma(t)}{m^2(t)} - \beta \text{tr}[\sigma(t) \sigma(t)] S^{-1}(t) \Phi(t) \right] \]
\[ \leq \beta \text{tr}[\Psi(t) \sigma(t) \sigma(t) S^{-1}(t) \Phi(t)]. \]
(64)

Lettng \( \chi(t) = \text{diag}\{\gamma_1^2 \delta_1(t) \delta_1(t), \gamma_2^2 \delta_2(t) \delta_2(t), \ldots, \gamma_M^2 \delta_M(t) \delta_M(t)\} \)
and substituting (64)-(65) to (63), by (57), we have
\[ V(\tilde{\Psi}_j(t+1), \tilde{\Phi}(t+1)) - V(\tilde{\Psi}_j(t), \tilde{\Phi}(t)) \]
\[ = - \left[ \frac{\epsilon(t) \sigma(t) \sigma(t)}{m^2(t)} - \beta \text{tr}[\sigma(t) \sigma(t)] S^{-1}(t) \Phi(t) \right] \]
\[ \leq \frac{\epsilon(t) \sigma(t) \sigma(t)}{m^2(t)} \chi(t) \]
\[ \leq \frac{\epsilon(t) \sigma(t) \sigma(t)}{m^2(t)} \chi(t). \]
(67)

Since \( S^{-1} \) is a positive definite matrix, there exists an orthogonal matrix \( T_s \in \mathbb{R}^{M \times M} \) such that \( T_s^T T_s = I \) and \( T_s^T S^{-1} T_s = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_M\} \) with \( \lambda_j \) denoting the eigenvalues of \( S^{-1} \). Then, from (60) and (66), we have
\[ 2S^{-1} - \chi(t) + \beta S^{-1} \sigma(t) \sigma(t) \leq \frac{\epsilon(t) \sigma(t) \sigma(t)}{m^2(t)} \chi(t) \]
\[ \leq \frac{\epsilon(t) \sigma(t) \sigma(t)}{m^2(t)} \chi(t) \]
\[ \leq \frac{\epsilon(t) \sigma(t) \sigma(t)}{m^2(t)} \chi(t). \]
(68)
which follows from the definitions of \( \gamma_j \) and \( \beta \) below (60) that
\[ 2S^{-1} - \chi(t) + \beta S^{-1} \sigma(t) \sigma(t) \geq \alpha_0 I \]
(69)
with some \( \alpha_0 > 0 \). Combining (67) and (69) reveals
\[ \text{tr}[\tilde{\Psi}_j(t+1) - \tilde{\Psi}_j(t)] - \text{tr}[\tilde{\Phi}(t+1) - \tilde{\Phi}(t)] \]
\[ \leq \frac{\epsilon_t^2}{m^2(t)} \]
(70)
which implies that \( V \) is non-increasing and \( \frac{\epsilon(t)}{m(t)} \in L^2 \cap L^\infty \)

Thus, \( \tilde{\Psi}_j(t) \in L^\infty, \Phi(t) \in L^\infty \), that is, \( \Psi_j(t) \in L^\infty, \Phi(t) \in L^\infty \). From (58) and (59), the property \( \frac{\epsilon(t)}{m(t)} \in L^2 \cap L^\infty \) yields \( \Psi_j(t+1) - \Psi_j(t) \in L^2 \cap L^\infty \) and \( \Phi(t+1) - \Phi(t) \in L^2 \cap L^\infty \). Due to \( \frac{\epsilon(t)}{m(t)} \), \( \Psi_j(t+1) - \Psi_j(t) \in L^\infty \) and \( \Phi(t+1) - \Phi(t) \in L^\infty \) all belonging to \( L^2 \), the limitation properties specified in (iii) of Lemma 2 hold.

This lemma shows that the parameter update laws ensure some desired properties of the parameter estimates. Note that although \( \Psi_j(t+1) - \Psi_j(t) \) and \( \Phi(t+1) - \Phi(t) \) converge to zero asymptotically, it does not assure the convergence of \( \Psi_j(t) \) and \( \Phi(t) \). Next, with the adaptive control law to hand, we will analyze the closed-loop system performance.
F. Stability Analysis

Before giving the main result, we first specify a useful lemma which is crucial for stability analysis. To proceed, we first introduce some useful notation:
- \( c \) denotes a signal bound;
- \( \tau(t) \) denotes a generic \( L^2 \cap L^\infty \) function that goes to zero as \( t \to \infty \);
- \( L^\infty \) denotes the set \( \{ r(t) | r(t) \in L^\infty \} \), for any function \( r(t) \) with \( r_s(t) = r(t), \ t \leq s, \) and \( r_s(t) = 0, \ t > s \).

Now, we give the following lemma.

**Lemma 3:** For three discrete-time signals \( r_i(t) \in \mathbb{R}^{p_i}, \ i = 1, 2, 3, \) with \( p_1 = p_2 \), such that \( r_1(t) = h(z)[r_2(t), r_2(t) \in L^\infty, \ r_3(t) \in L^\infty, \) if \( ||r_2(t)|| \leq \tau_1(t) \sup_{k \leq t} ||r_3(k)|| + \tau_2(t), \ \forall t \geq 0, \) then
\[
||r_1(t)|| \leq \tau_3(t) \sup_{k \leq t} ||r_3(k)|| + \tau_4(t), \ \forall t \geq 0,
\]
where \( \tau_i(t), \ i = 1, 2, 3, 4, \) are all \( L^2 \cap L^\infty \) functions.

**Proof:** Let \( h_z(t) \) denote the impulse response function of \( h(z) \). From (50) and (51), it can be verified that
\[
h_z(t) = \frac{1}{\alpha} (\alpha^t - \delta(t)) \geq 0, \ \forall t \geq 0,
\]
where \( \delta(t) \) denotes the unit impulse response. Then, with \( r_1(t) = h(z)[r_2(t), r_2(t) \in L^\infty, \ r_3(t) \in L^\infty, \) if \( ||r_2(t)|| \leq \tau_1(t) \sup_{k \leq t} ||r_3(k)|| + \tau_2(t), \ \forall t \geq 0, \) then
\[
||r_1(t)|| \leq \tau_3(t) \sup_{k \leq t} ||r_3(k)|| + \tau_4(t), \ \forall t \geq 0,
\]
where \( \tau_i(t), \ i = 1, 2, 3, 4, \) are all \( L^2 \cap L^\infty \) functions.

**Theorem 1:** Under Assumptions 1-2, the adaptive control law (28) with the parameter update laws (58) and (59), applied to the system (1) with vector relative degree \( [1, 1, \ldots, 1] \) and unknown \( B, C, \Theta_f, \) ensures closed-loop stability and asymptotic output tracking:
\[
\lim_{t \to \infty} ||y(t) - y^*(t)|| = 0.
\]

**Proof:** The proof contains four steps.

**Step 1:** Show \( \sum_{j=1}^M |\sigma_j(t)| \leq \tau \sup_{k \leq t} ||e(k)|| + \tau. \) Based on the fact that \( ||y^*(t)|| \leq c (\tau \) and \( c \) were defined above

Lemma 3) and \( f_{ik}(x(t)) \) are globally Lipschitz in \( x(t) \), we derive from (46)-(49) that
\[
||\varphi_j(t)|| \leq ||\varphi_j(x(t))|| + ||e(t)|| + c \leq c||\xi(t)|| + c||\eta(t)|| + c||e(t)|| + c
\]
which, together with (19), (31), and \( \xi(t) = y(t) \), implies
\[
||\varphi_j(t)|| \leq c||e(t)|| + c.
\]
In particular, since \( \frac{z}{h_0(z)} \) is proper and stable,
\[
||\frac{z}{h_0(z)}|| \leq c \sup_{k \leq t} ||e(k)|| + c.
\]
From (50), (51), (55), and (56), we have
\[
h_0(z)[\sigma_j(t)] = h_0(z)[\Psi_j(t) - \varphi_j(t) = \Psi_j(t) + 1 - \varphi_j(t) - \varphi_j(t) = \Psi_j(t) - \varphi_j(t) = \frac{z}{h_0(z)} \varphi_j(t)]
\]
which reveals that \( \sigma_j(t) \) can be expressed as
\[
\sigma_j(t) = h(z)[\frac{z}{h_0(z)} \varphi_j(t)]
\]
\[\text{Note that}
||\frac{z}{h_0(z)}|| \varphi_j(t) \leq c \sup_{k \leq t} ||e(k)|| + c.
\]
From (80) and (82), using Lemma 3, we obtain
\[
|\sigma_j(t)| \leq \tau \sup_{k \leq t} ||e(k)|| + \tau, \ \forall j = 1, 2, \ldots, M,
\]
which implies that
\[
\sum_{j=1}^M |\sigma_j(t)| \leq \tau \sup_{k \leq t} ||e(k)|| + \tau.
\]

**Step 2:** Show \( m(t) \leq \sup_{k \leq t} ||e(k)|| + c. \) From the definition of \( \delta_j(t) \) in (56), with the fact that \( h(z) \) is strictly proper and stable, we obtain \( ||\delta_j(t)|| \leq \sup_{k \leq t} ||e(k)|| + c, \) which follows from (77) that
\[
||\delta_j(t)|| \leq \sup_{k \leq t} ||e(k)|| + c.
\]
From the definition of \( m(t) \) in (60), we derive
\[
m(t) \leq 1 + \sum_{j=1}^M |\sigma_j(t)| + \sum_{j=1}^M ||\delta_j(t)||
\]
In addition to (84) and (85), the inequality (86) yields
\[
m(t) \leq \sup_{k \leq t} ||e(k)|| + c.
\]
Step 3: Show $e(t) \in L^\infty$. From (54), we get
\[ ||e(t)|| \leq |e(t)| + ||\varphi(t)|| \leq m(t) \frac{|e(t)|}{m(t)} + ||\varphi(t)||. \] (88)
With the fact that $\frac{e(t)}{m(t)} \in L^2 \cap L^\infty$, $\varphi(t) \in L^\infty$, and (84), the inequality (88) reveals
\[ ||e(t)|| \leq \tau \sup_{k \leq t} ||e(k)|| + \tau \] (89)
which further implies that
\[ \sup_{k \leq t} ||e(k)|| \leq \tau \sup_{k \leq t} ||e(k)|| + \tau. \] (90)
From the definition of $\bar{e}(t)$ in (52), we have
\[ e(t) = \frac{P_m^{-1}(z)}{h_0(z)} \bar{e}(t), \] (91)
where $P_m^{-1}(z)$ is the inverse of $P_m(z)$. Note that $\frac{P_m^{-1}(z)}{h_0(z)}$ is proper and stable. Thus, (90) and (91) derive that
\[ ||e(t)|| \leq c \sup_{k \leq t} ||e(k)|| + c \leq \tau \sup_{k \leq t} ||e(k)|| + c \] (92)
which implies $e(t) \in L^\infty$.

Step 4: Show closed-loop stability and $\lim_{t \to \infty} (y(t) - y^*(t)) = 0$. Since $e(t) \in L^\infty$, it follows from (89) that $\bar{e}(t) \in L^2 \cap L^\infty$, and in turn from (91) that $e(t) \in L^2 \cap L^\infty$. Thus,
\[ \lim_{t \to \infty} (y(t) - y^*(t)) = 0. \] (93)
Due to the boundedness of $e(t)$, from (77), (84), (85), and (87), we obtain the boundedness of $\varphi_j(t), \sigma_j(t), \delta_j(t)$, and $m(t)$, respectively. Moreover, from $\xi(t) = y(t)$, we derive the boundedness of $\xi(t)$, which follows from (19) that $\eta(t) \in L^\infty$. Thus, based on $T'(x) = [\xi^T, \eta^T]^T$, we have the boundedness of $x(t)$. Finally, from (28), we have the boundedness of $u(t)$. Therefore, all closed-loop signals are bounded.

So far, we have developed a linear parameterization based adaptive control scheme for the system (1) with vector relative degree $[1,1,...,1]$. The proposed adaptive control scheme not only ensures the adaptive control gain matrix is always nonsingular, but also guarantees desired system performance: closed-loop stability and asymptotic output tracking.

Remark 8: For adaptive control of MIMO discrete-time nonlinear systems, this paper is the first to address the singularity problem of the adaptive high frequency gain matrix. It is also the first to solve this problem based on a linear parameterization based framework. The proposed adaptive control scheme does not require that the high frequency gain matrix is positive or negative definite (which is a common design condition for CT case in the literature).

G. An Extension
In this part, we will show that the adaptive control scheme proposed above is applicable to adaptive control of canonical-form MIMO DT nonlinear systems with a general vector relative degree. A brief outline for the adaptive control design will be given.

System model. We consider the following canonical-form MIMO DT nonlinear system
\[ y_i(t + \rho_i) = f_i(x(t)) + \sum_{j=1}^{M} g_{ij}(x(t)) u_j(t), i = 1, 2, ..., M, \]
\[ \eta(t + 1) = q(\xi(t), \eta(t), u(t)), \] (94)
where the state vector, input vector, and output vector are defined as
\[ x(t) = [y_1(t), ..., y_1(t + \rho_1 - 1), ..., y_M(t), ..., y_M(t + \rho_M - 1)]^T \in \mathbb{R}^L, \]
\[ u(t) = [u_1(t), u_2(t), ..., u_M(t)]^T \in \mathbb{R}^M, \]
\[ y(t) = [y_1(t), y_2(t), ..., y_M(t)]^T \in \mathbb{R}^M, \] (95)
respectively, with $\eta \in \mathbb{R}^r$ and $L = \sum_{i=1}^{M} \rho_i + r$; and $f_i \in \mathbb{R}, g_{ij} \in \mathbb{R}, i, j = 1, 2, ..., M$, are globally Lipschitz functions with linearly parametrized uncertainties. We assume that the state system $x(t)$ is measurable, which implies that $y_i(t + k), i = 1, 2, ..., M, k = 1, ..., \rho_i - 1$, can be used for adaptive control law design. Note that the controlled plant (94) is in a canonical-form with vector relative degree $[\rho_1, \rho_2, ..., \rho_M]$.

Assumptions. The conditions needed for adaptive control design of the system (94) are (i) the internal dynamics need to be ISS, which is similar to Assumption 1; (ii) the system (94) has a well-defined vector relative degree, that is, the matrix $\{g_{ij}(x)\}, i, j = 1, 2, ..., M$, is nonsingular for all $x \in \mathbb{R}^n$; and (iii) the matrix $\{g_{ij}(x)\}$ can be decomposed into the form $\Theta_\varphi^* \Phi(x)$ with unknown constant square matrix $\Theta_\varphi^*$ and known time-varying square matrix $\Phi(x)$ such that $||\Phi(x)||$ is bounded and away from zero.

Next, we show how to proceed the adaptive control design based on the proposed adaptive control scheme.

Parametrized model. Following the proposed adaptive control scheme, the output dynamics are expressed as
\[ \begin{bmatrix} y_1(t + \rho_1) \\ y_2(t + \rho_2) \\ \vdots \\ y_M(t + \rho_M) \end{bmatrix} = \Theta_{\varphi}^* \varphi_\varphi(x(t)) + S^* D_x U_s \bar{u}(t), \] (98)
where $\Theta_{\varphi}^* \varphi_\varphi(x(t))$ is a parametrized model of the vector $[f_1(x(t)), f_2(x(t)), ..., f_M(x(t))]^T$. $S^* D_x U_s$ is a decomposition of $\Theta_\varphi^*$, and $\bar{u}(t) = \Phi(x(t)) u(t)$. Based on the assumptions (ii) and (iii), we see that $\Phi(x(t))$ is always nonsingular with a bounded inverse for all $x \in \mathbb{R}^n$. Thus, as long as $\bar{u}(t)$ is chosen, the control law $u(t)$ can be calculated as $u(t) = \Phi^{-1}(x(t)) \bar{u}(t)$.

Now, from (98), we derive the parametrized model as
\[ \begin{bmatrix} y_1(t + \rho_1) \\ \vdots \\ y_M(t + \rho_M) \end{bmatrix} = D_x \Theta_{\varphi}^* \varphi_\varphi(x(t)) + D_x \Theta_{\varphi}^* \bar{u}(t) + D_x \bar{u}(t), \] (99)
where $\Theta_1^*$ and $\Theta_2^*$ have the same expressions as those $\Theta_1^s$ and $\Theta_2^s$ of the vector relative degree $[1, 1, \ldots, 1]$ case.

Adaptive control law. Motivated by (28), with $y^*(t) = (y_1^*(t), \ldots, y_M^*(t))^T$, the adaptive control law is designed as

$$u(t) = \Phi_y^1(x(t))\hat{u}(t),$$

$$\hat{u}(t) = -\Theta_2(t)u(t) - \Theta_1(t)\phi_f(x(t)) + \Theta_3(t)y^*(t + \rho) - \Theta_3(t)$$

$$= \sum_{i=0}^{\rho-1} a_i(y_1(t + i) - y_1^*(t + i))$$

(100)

where $\Theta_1(t), i = 1, 2, 3$, have the same expressions as those $\Theta_1(t)$ of the vector relative degree $[1, 1, \ldots, 1]$ case. In particular, $a_{j_1}$ are chosen constant parameters such that all zeros of $P_{mi}(z), i = 1, 2, \ldots, M$, are inside the unit circle of the complex $z$-plane, where $P_{mi}(z)$ are defined as

$$P_{mi}(z) = z^m + a_{\rho}z^{m-1} + \cdots + a_1z + a_0.$$ (101)

One can verify that the particular structure of the adaptive control law ensures the adaptive control law is always non-singular in the process of parameter adaptation. Note that the adaptive control law contains $y(t + i), i = 1, 2, \ldots, M - 1$, which are part of the system state vector and surely available at the current time instant.

Tracking error model. Let $e(t) = y(t) - y^*(t)$. Substituting (100) to (99) gives

$$S^{s-1} = \begin{bmatrix}
e_1(t + \rho_1) + \sum_{i=0}^{\rho_1-1} a_1e_1(t + i) \\
\vdots \\
e_m(t + \rho M) + \sum_{i=0}^{\rho M-1} a_Me_M(t + i) \end{bmatrix}$$

(102)

where $\Theta_3(t)$ and $\Theta_3^*(t)$ have the same expressions as those $\Theta_3^s(t)$ and $\Theta_3^s(t)$ of the vector relative degree $[1, 1, \ldots, 1]$ case. Define

$$P_{m}(z) = \text{diag}(P_{mi}(z)), \quad \bar{T}(t) = (\Theta_1(t), \Theta_2(t), \Theta_3(t)), (103)$$

$$\tilde{\Phi}(t) = \Psi(t) - \Psi^*(t) = (\bar{T}(t), \tilde{T}(t), \bar{T}(t)), (104)$$

$$\varphi(t) = [-\varphi^1_1(x(t)), -\tilde{u}^2(t), -\sum_{i=0}^{\rho_1-1} a_1e_1(t + i) + y_1^*(t + \rho_1), \ldots, -\sum_{i=0}^{\rho_M-1} a_Me_M(t + i) + y_M^*(t + \rho M)]^T.$$ (105)

Then, (102) can be expressed as

$$P_{m}(z)[e(t)] = S^*D_{\bar{T}}(t)\varphi(t)$$

(106)

which is the expected tracking error model.

With the adaptive control law (100) and the tracking error model (106), following the design procedure of the vector relative degree $[1, 1, \ldots, 1]$ case, we can sequentially derive the corresponding estimation errors, parameter update laws, and finally prove that the adaptive control law (100) can ensure closed-loop stability and asymptotic output tracking for the system (94). Here, we do not provide the details which contain a lot of expressions and computation.

So far, we have developed a matrix decomposition based adaptive control scheme for the system (1) with vector relative degree $[1, 1, \ldots, 1]$, and shown that the control scheme is applicable to adaptive control of canonical-form MIMO DT systems with a general vector relative degree.

V. NOMINAL CONTROL DESIGN FOR HIGH-ORDER VECTOR RELATIVE DEGREE CASE

A nominal control framework is presented for the system (1) with a high-order vector relative degree. To make the derivations of this section readable, as a representative case, we present a detailed control design procedure for the system (1) with vector relative degree $[2, 2, \ldots, 2]$, which can be referred to extend to a general vector relative degree case $[\rho_1, \rho_2, \ldots, \rho_M]$ with $\rho_i > 1$.

A. System Model and Design Conditions

For the system (1) with vector relative degree $[2, 2, \ldots, 2]$, from (12), the output dynamics are

$$y(t + 2) = \Theta_{ef}^s\phi_f(x(t + 1)) = \Theta_{ef}^s\phi_f(\Theta_{ef}^s\phi_f(x(t)) + Bu(t))$$

(107)

such that $\Theta_{ef}^s\frac{\partial \phi_f}{\partial \phi_f}(\Theta_{ef}^s\phi_f(x(t)) + Bu(t))$ is nonsingular for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^M$.

From the structure of (107), we see that the output dynamics contain the linearly parametrized uncertainty $\Theta_{ef}^s$, the nonlinearly parametrized uncertainty $\Theta_f$, and the control input $u(t)$ of a non-affine form. To handle the high-frequency gain $\Theta_{ef}^s\frac{\partial \phi_f}{\partial \phi_f}(\Theta_{ef}^s\phi_f(x(t)) + Bu(t))$, we need some corresponding design conditions. Thus, letting $[X]_{ij}$ denote the $ij$-th entry of any matrix $X$, we make the following two assumptions.

Assumption 3: For $i, j = 1, 2, \ldots, M$,

$$\left|\Theta_{ef}^s\frac{\partial \phi_f}{\partial \phi_f}(\Theta_{ef}^s\phi_f(x(t)) + Bu(t))\right|_{ij} \geq \varepsilon_0$$

(108)

with some constant $\varepsilon_0 > 0$, and the signs of $\Theta_{ef}^s\frac{\partial \phi_f}{\partial \phi_f}(\Theta_{ef}^s\phi_f(x(t)) + Bu(t))$ are known.

Remark 9: The property shown in (108) implies that the gain $\Theta_{ef}^s\frac{\partial \phi_f}{\partial \phi_f}(\Theta_{ef}^s\phi_f(x(t)) + Bu(t))$ is strictly diagonal dominant. Note that the strictly diagonal dominance of control gains is a commonly used design condition for adaptive control design of MIMO nonlinear systems with practical applications in the literature ([27]-[29]). In comparison, for adaptive control design of SISO nonlinear systems, a commonly used design condition on the high frequency gain (denoted as $k_p(x)$) is $|k_p(x)| \geq c_0$ for some constant $c_0 > 0$. The condition (108) used for MIMO nonlinear systems can be seen as an extension of $|k_p(x)| \geq c_0$ used for SISO nonlinear systems. Note that this paper does not consider the unknown sign case, which needs further study.

Assumption 4: There exists some decomposition of $\Theta_{ef}^s$, denoted as $\Theta_{ef}^s = K^*_pK_q$, such that $K^*_p \in \mathbb{R}^{M \times M}$ is unknown.
and $K_p \in \mathbb{R}^{M \times \sum_{i=1}^n p_i}$ is known. In addition, all leading principal minors of $K_p^*$, defined as $\Delta_i$, $i = 1, 2, ..., M$, are nonzero and their signs are known, that is, $\Delta_i \neq 0$ and $\text{sign}(\Delta_i)$ are known.

**Remark 10:** Assumption 4 means that some information about the parameters and structural properties of $\Theta^*_f$ is known, such that, based on such information, one can decompose it into a product of an unknown matrix and a known matrix. Based on Assumption 4, $K_p^*$ can be decomposed as $K_p^* = S^* D_s U_s$, where $S^*$ is a positive definite matrix, $U_s$ is still a unit upper triangular matrix, and

$$D_s = \text{diag}\{s_1^*, s_2^*, ..., s_M^*\}$$

$$= \text{diag}\{\text{sign}[d_1^*]\gamma_1, \text{sign}[d_2^*]\gamma_2, ..., \text{sign}[d_M^*]\gamma_M\} \quad (109)$$

such that $\gamma_i > 0$, $i = 1, ..., M$, may be arbitrary.

Due to the existence of the nonlinear uncertainty $\Theta_f$ and the non-affine control input $u(t)$ in $y(t + 2)$, we cannot derive a parametrized model. Instead, under Assumptions 3 and 4, we have

$$y(t + 2) = S^* D_s U_s K_q \phi_f(\Theta^*_f \phi_f(x(t)) + Bu(t)) \quad (110)$$

which yields a matrix decomposition based model of the output dynamics of the form

$$S^{-1} y(t + 2) = D_s \Theta^*_f K_q \phi_f(\Theta^*_f \phi_f(x(t)) + Bu(t)) + D_s K_q \phi_f(\Theta^*_f \phi_f(x(t)) + Bu(t)) - S^{-1} y(t + 2) \quad (111)$$

where $\Theta^*_f$ has the same expression as $\Theta^*_f$ of the vector relative degree $[1, 1, ..., 1]$ case.

**B. Nominal Control Law**

Introduce an implicit function with respect to $u$ defined as

$$H_0(x(t), u, y^*(t + 2)) = D_s \Theta^*_f K_q \phi_f(\Theta^*_f \phi_f(x(t)) + Bu(t)) + D_s K_q \phi_f(\Theta^*_f \phi_f(x(t)) + Bu(t)) - S^{-1} y(t + 2). \quad (112)$$

Based on Assumption 3, one can verify that, for $i, j = 1, 2, ..., M$,

$$\left\lfloor \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} \right\rfloor_{ii} = \left\lfloor \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} \right\rfloor_{i j} \quad (113)$$

for some constant $\varepsilon_1 > 0$, and the signs of $\left\lfloor \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} \right\rfloor_{ii}$ are known. Without loss of generality, we assume that

$$\left\lfloor \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} \right\rfloor_{ii} > 0 \quad (114)$$

for $i = 1, 2, ..., k$, and

$$\left\lfloor \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} \right\rfloor_{ii} < 0 \quad (115)$$

for $i = k + 1, ..., M$. Then, let

$$H(x(t), u, y^*(t + 2)) = [H_{01}, ..., H_{0k}, -H_{0k+1}, ..., -H_{0M}]^T, \quad (116)$$

where $H_{0i}$, $i = 1, 2, ..., M$, denote the $i$-th element of $H_0(x(t), u, y^*(t + 2))$. Then,

$$\left\lfloor \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} \right\rfloor_{ii} = \left\lfloor \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} \right\rfloor_{i j} \quad (117)$$

$$\left\lfloor \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} \right\rfloor_{ij} = \left\lfloor \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} \right\rfloor_{ij}, \quad i \neq j. \quad (118)$$

As $\phi_f$ is continuously differentiable, $H(x, u, y^*)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^{M} \times \mathbb{R}^{M}$. The nominal control law is chosen as the solution to the equation

$$H(x(t), u, y^*(t + 2)) = 0 \quad (119)$$

with respect to $u$, which can be expressed as

$$u = g(x(t), y^*(t + 2)). \quad (120)$$

We will show that the nominal control law (120) is unique in the sense that the equation (119) has a unique solution $u$.

**C. Stability Analysis**

With the help of the implicit function result developed in [32], we derive the following result.

**Theorem 2:** Under Assumptions 1, 3 and 4, there exists a unique solution to (119) with respect to $u$, and the solution is the nominal control law, applied to the system (1) with vector relative degree $[2, 2, ..., 2]$, ensures closed-loop stability and

$$y(t + 2) - y^*(t + 2) = 0. \quad (121)$$

**Proof:** We first show that (119) has a unique solution, and then, analyze output tracking and close-loop stability.

**Step 1:** Construct a contraction mapping. With $H$ defined in (116), at each time $t$, let

$$\vartheta = H(x(t), 0, y^*(t + 2)). \quad (122)$$

If $\vartheta = 0$, then $u$ is the solution to (119). Otherwise, with $X = \{X_1, X_2, ..., X_p\} \subset \mathbb{R}^{p \times q}$ and $\|X\|_\varnothing = \max \{\sum_{i=1}^{q} |X_i|, \sum_{j=1}^{p} |X_2|, ..., \sum_{k=1}^{q} |X_p|\}$, we consider a compact set

$$\Omega_k = \{u||u||_\varnothing \leq \frac{\|\vartheta\|_\varnothing}{\varepsilon_0}\}. \quad (123)$$

Then, at each time $t$, fixing $x(t)$ and $y^*(t + 2)$, it follows from (108) and (112) that there exist two positive constants $D_t$ and $d_t$ such that, for all $u \in \Omega_k$ and $i, j = 1, 2, ..., M$,

$$\left\lfloor \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} \right\rfloor_{ii} \leq D_t, \quad (124)$$

$$\left\lfloor \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} \right\rfloor_{ij} \geq d_t. \quad (125)$$
Introduce a mapping $f_l: \Omega_l \rightarrow \mathbb{R}^M$ defined as
$$f_l(u) = u - \frac{1}{l} H(x(t), u, y^*(t + 2)),$$
where $l$ is a chosen constant such that $l > D_l$.

Now, we prove that $f_l$ is a contraction mapping. For each $u \in \Omega_l$, by Mean Value Theorem, we obtain
$$H(x(t), u, y^*(t + 2)) = \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} |_{u = \sigma_1, u} + H(x(t), 0, y^*(t + 2)),$$
where $\sigma_1 \in \mathbb{R}^M$ is some vector fulfilling $\|\sigma_1\|_\infty \leq \|u\|_\infty$ which implies that $\sigma_1 \in \Omega_l$. Then, (126) and (127) yield that
$$\| f_l(u) \|_\infty = \left\| u - \frac{1}{l} \left( \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} |_{u = \sigma_1, u} + H(x(t), 0, y^*(t + 2)) \right) \right\|_\infty \leq \frac{1}{l} \left\| \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} |_{u = \sigma_1, u} \right\|_\infty + \frac{\| H(x(t), 0, y^*(t + 2)) \|_\infty}{l},$$
where $I \in \mathbb{R}^{M \times M}$ is an identical matrix. Since $l > D_l$, from (125), we have $1 - \frac{1}{l} \left[ \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} |_{u = \sigma_1, u} \right] > 0$. Thus, with the fact that $l > D_l \geq \varepsilon_0$, we have
$$\left\| I - \frac{1}{l} \left[ \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} |_{u = \sigma_1, u} \right] \right\|_\infty = \max_{i=1, \ldots, M} \left\{ 1 - \frac{1}{l} \right\} \left[ \left[ \frac{\partial H(x(t), u, y^*(t + 2))}{\partial u} |_{u = \sigma_1, u} \right] \right\}_i \leq \frac{\| u \|_\infty}{\varepsilon_0} \leq \frac{\| u \|_\infty}{\varepsilon_0},$$
which implies that $f_l$ maps $\Omega_l$ into itself. Moreover, for any $u_1, u_2 \in \Omega_l$, we have $\| f_l(u_1) - f_l(u_2) \|_\infty \leq \| u_1 - u_2 \|_\infty - \frac{1}{l} (H_{u=u_1} - H_{u=u_2}) \|_\infty$ which follows from Mean Value Theorem that
$$\| f_l(u_1) - f_l(u_2) \|_\infty \leq \left\| I - \frac{1}{l} \frac{\partial H}{\partial u} |_{u=u_2} \right\|_\infty \| u_1 - u_2 \|_\infty \leq \frac{1}{l} \| u_1 - u_2 \|_\infty,$$
where $\sigma_2 \in \Omega_l$, which implies that $f_l$ is a contraction mapping.

Step 2: Show output tracking and closed-loop stability. Based on Banach’s Fixed Point Theorem, there exists a unique solution to $f_l(u) = u$, that is,
$$u - H(x(t), u, y^*(t + 2)) = u$$
has a unique solution denoted as $u^*_t$. Thus, $H(x(t), u^*_t, y^*(t + 2)) = 0$ always holds at each time $t$. From (111) and (112), the equation $H(x(t), u^*_t, y^*(t + 2)) = 0$ implies that $S^{*-1} y(t + 2) - S^{*-1} y^*(t + 2) = 0$ which deduces to
$$y(t + 2) - y^*(t + 2) = 0.$$

Since $y^*(t)$ is bounded, we have $\varepsilon(t) = [y_1(t), y_1(t + 1), \ldots, y_M(t), y_M(t + 1)]^T$ is bounded. Under Assumption 1, the boundedness of $\varepsilon(t)$ implies that of $\eta(t)$, which follows from the diffeomorphism $\hat{T}(x) = [\xi^T, \eta^T]^T$ that $x(t)$ is bounded. As $u^*_t$ belongs to $\Omega_t$, we obtain the control law is also bounded. Thus, all closed-loop signals are bounded. \( \nabla \)

Theorem 2 is the fundamental for adaptive control design, which reveals that there exists an ideal control law that can ensure closed-loop stability and exact output tracking. So far, we have developed a nominal control scheme for the system (1) with vector relative degree $[2, 2, \ldots, 2]$.

Remark 11: To derive the nominal control law, a natural way, in practice, is to solve the equation (119) to get an analytical solution $u(t)$. As $u(t)$ nonlinearly exists in (119), it may be difficult to obtain. Thus, as alternative, we can design an iteration based control law defined by
$$u_i(t) = u_{i-1}(t) - \frac{1}{l} H(x(t), u(t-1), y^*(t + 2)), i = 1, 2, \ldots, \text{(134)}$$
with $u_0(t) = u(t-1)$, where $u(t-1)$ denotes the control law chosen at the former time instant $t-1$. Following an analysis similar to the proof of Theorem 2, one can verify that $\{u_i(t)\}$ is convergent to $u^*_t$ for all $t = 1, 2, \ldots$. Here, we do not provide further details for the iteration based control law.

Remark 12: Motivated by the nominal control design, if we can derive an estimate of $H(x(t), u(t), y^*(t + 2))$, denoted as $\hat{H}(x(t), u(t), y^*(t + 2), \Theta(t))$ with $\Theta(t)$ denoting the set of the estimates of unknown parameters, such that
$$\frac{\partial H(x(t), u(t), y^*(t + 2), \Theta(t))}{\partial u} \mid_{u=\hat{u}} \geq \varepsilon_2$$
for some constant $\varepsilon_2 > 0$, then we are able to follow the construction of $f_l(u)$ in (126) to construct a contraction mapping for the adaptive control case, based on which an adaptive control law could be derived. Thus, the key point is to derive the estimates of $\Theta^*_t, \Theta_t^*, B, S^{*-1}$, which should be suitable for constructing $\hat{H}(x(t), u(t), y^*(t + 2), \Theta(t))$. Such a work is not simple and needs a lot of time and effort, which is left as a future study.

VI. SIMULATION STUDY

This section presents a representative example to show the design procedure and verify the effectiveness of the proposed control method.

A. System Model

Consider the following numerical system model
$$x(t + 1) = f(x(t)) + B_1 u_1(t) + B_2 u_2(t),$$
$$y_j(t) = C_j x(t), \quad j = 1, 2,$$
\text{(135)}
where $x(t) = [x_1(t), x_2(t), x_3(t)] \in \mathbb{R}^3$ is the system state, $u_j, y_j$, $j = 1, 2$, are the control input variables and the output variables, respectively, $B_1 = [1, 0, 1]^T$, $B_2 = [1, 2, 0]^T$, $C_1 = [-1, 0, 0]$, $C_2 = [1, -2, -1]$, and $f(x(t)) = \left[ f_1(x(t)), f_2(x(t)), f_3(x(t)) \right]^T$ with

\[
f_1(x) = \theta_1^T \phi_1(x) = [1.2, 2.08, 0.24][x_1, x_1 \sin x_3, \sqrt{1 + x_2^2}]^T,
\]

where $\theta_1 = [\theta_1^T, \phi_1^T]^T$.

The simulation assumes that $B_j, C_j, \theta_2^T$, $i = 1, 2, 3$, $j = 1, 2$, are all unknown, and $\phi_j, i = 1, 2, 3$, are known.

B. System Transformation

From the parameters in $B_j, C_j, \theta_2^T$, $i = 1, 2, 3$, we obtain

\[
\begin{bmatrix}
C_1 B_1 & C_1 B_2 \\
C_2 B_1 & C_2 B_2
\end{bmatrix} = \begin{bmatrix}
-1 & -1 \\
0 & -3
\end{bmatrix}
\]

which implies that the system model (135) has vector relative degree $[1, 1]$. Then, based on Lemma 1, letting $\xi(t), \eta(t)^T = [y_1(t), y_2(t)]^T$ and $\eta(t) = x_3(t)$, the system model is transformed into two subsystems: the output dynamics

\[
y_1(t + 1) = -f_1(x(t)) - u_1(t) - u_2(t),
\]

\[
y_2(t + 1) = f_1(x(t)) - 2f_2(x(t)) - f_3(x(t)) - 3u_2(t),
\]

and the internal dynamics $\eta(t + 1) = f_3(x(t)) + u_1(t)$.

C. Verification of Design Conditions

By using the simulation example (135), we explain how to verify Assumptions 1 and 2.

It follows from the structure of $f_i(x(t))$ defined in (136)-(138) that $f_i(x(t))$, $i = 1, 2, 3$, are Lipschitz in $x(t)$, which implies that $\phi_f(x(t))$ defined in (4) is Lipschitz in $x(t)$. From the structure of the adaptive control law (28), one can verify that $u(t)$ can be expressed as $u(t) = f_1(x(t)) + f_2(x(t))y(t + 1) + f_3(x(t))$, where $f_1, f_2, f_3$ are bounded and can be specified based on (147). Then, we derive that $u(t)$ is Lipschitz in $\phi_f(x(t)), y(t + 1), y(t)$, which further implies that $u(t)$ is Lipschitz in $x(t), y(t + 1), y(t)$. Therefore, $f_3(x(t)) + u_1(t)$ is Lipschitz in $\xi(t)$ and $y(t + 1)$.

Then, if one can identify the signs of $C_1 B_1$ and $C_2 B_2$, Assumption 2 can be verified. Some parameter identification algorithms can be applied to acquire the signs of $C_1 B_1$ and $C_2 B_2$. A number of parameter identification algorithms can handle this issue. The parameter identification issue is out of the scope of this paper, and we do not provide further details.

D. Parametrized Model

With the relative degree condition, the output dynamics (140) is parametrized as

\[
y(t + 1) = \Theta_f \phi_f(x(t)) + \Theta_b u(t),
\]

where

\[
\Theta_f = \begin{bmatrix}
-1.2 & -2.08 & -0.24 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
\Theta_b = \begin{bmatrix}
-1 & -1 & 0 & -3
\end{bmatrix}.
\]

To derive the parametrized model, we first decompose $\{C_iB_i\}$ as the SDU form

\[
\begin{bmatrix}
C_1 B_1 & C_1 B_2 \\
C_2 B_1 & C_2 B_2
\end{bmatrix} = S^*D_u U_s
\]

where $S^*, D_u, U_s$ are the three matrices of the right side of the equation, respectively. Then, from (25), the parametrized model of the output dynamics is

\[
S^{-1} y(t + 1) = D_s \Theta_f \phi_f(x(t)) + D_s \Theta_b u(t) + D_s u(t),
\]

and

\[
\Theta^*_f = \begin{bmatrix}
1 & 0 & -1 & 0 & 0 & 1 & 0 & 0
\end{bmatrix},
\]

\[
\Theta^*_b = \begin{bmatrix}
1.2 & 2.08 & 0.24 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

E. Adaptive Control Law

From (28), the adaptive control law is designed as

\[
u(t) = -\Theta_f \phi_f(x(t)) + \Theta_b u(t) + \Theta_f \phi_f(x(t)),
\]

where $y^* = [y_1^*, y_2^*]^T$ is chosen later, $\Theta_i(t), i = 1, 2, 3$, are the estimates of $\Theta^*_i$, $\Theta^*_2$, $(S^* D_u)^{-1}$, respectively, which have the forms

\[
\Theta_1(t) = \begin{bmatrix}
\theta_{111}(t) & \theta_{112}(t) & \cdots & \theta_{117}(t) \\
\theta_{121}(t) & \theta_{122}(t) & \cdots & \theta_{127}(t)
\end{bmatrix},
\]

\[
\Theta_2(t) = \begin{bmatrix}
\theta_{212}(t) \\
0 & 0
\end{bmatrix},
\]

\[
\Theta_3(t) = \begin{bmatrix}
\theta_{311}(t) & \theta_{312}(t) \\
\theta_{321}(t) & \theta_{322}(t)
\end{bmatrix},
\]

and

\[
(S^* D_u)^{-1} = \begin{bmatrix}
-1 & 0 & -0.3333
\end{bmatrix},
\]

Note that the control signal should be calculated as the procedure shown in (30).

F. Parameter Update Laws

Define

\[
\Psi(t) = \begin{bmatrix}
\theta_{111}(t) & \theta_{112}(t) & \cdots & \theta_{117}(t) \\
\theta_{121}(t) & \theta_{122}(t) & \cdots & \theta_{127}(t)
\end{bmatrix},
\]

\[
\varphi(t) = \begin{bmatrix}
\theta_{321}(t) & \theta_{322}(t) \\
\theta_{321}(t) & \theta_{322}(t)
\end{bmatrix} \in \mathbb{R}^{2 \times 11},
\]

\[
\varphi(t) = [-x_1, -x_1 \sin x_3, -\sqrt{1 + x_2^2}, -x_2 \sin x_1, -x_1,
\]

\[
\varphi(t) = \begin{bmatrix}
\varphi_{x_1} & \varphi_{x_2} & \varphi_{x_3} & \varphi_{x_4}
\end{bmatrix}
\]

\[
\varphi(t) = \begin{bmatrix}
\varphi_{x_1} & \varphi_{x_2} & \varphi_{x_3} & \varphi_{x_4}
\end{bmatrix}
\]
From (58)-(59), the parameter update where

\[ y(t + 1) \in \mathbb{R}^n, \]

\[ \Psi_1(t) = \begin{bmatrix} \theta_{110}(t), \theta_{111}(t), \theta_{112}(t), \theta_{311}(t), \theta_{312}(t) \end{bmatrix}^T \in \mathbb{R}^{15}, \]

\[ \Psi_2(t) = \begin{bmatrix} \theta_{121}(t), \theta_{122}(t), \theta_{127}(t), \theta_{221}(t), \theta_{222}(t) \end{bmatrix}^T \in \mathbb{R}^9, \]

\[ \varphi_1(t) = [-u_2(t), -x_1, -x_1 \sin x_3, -\sqrt{1 + x_2^2}, -x_2 \sin x_1, \]

\[ -x_1, -x_3, -\sin x_1, y_1(t + 1) - \frac{1}{2} y_1(t), \]

\[ y_2(t + 1) - \frac{1}{2} y_2(t)]^T \in \mathbb{R}^{10}, \]

\[ \varphi_2(t) = [-x_1, -x_1 \sin x_3, -\sqrt{1 + x_2^2}, -x_2 \sin x_1, -x_1, \]

\[ -x_3, -\sin x_1, y_1(t + 1) - \frac{1}{2} y_1(t), \]

\[ y_2(t + 1) - \frac{1}{2} y_2(t)]^T \in \mathbb{R}^{9}. \]

Letting \( h_0(z) = z - \frac{1}{z} \), we have \( h(z) = \frac{1}{z - \frac{1}{z}} \) whose impulse response function is \( h(t) \geq 0, \forall t \geq 0 \). Then,

\[ \epsilon(t) = \begin{bmatrix} z + \frac{1}{z} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} \left( \frac{1}{z - \frac{1}{z}} \right) \epsilon(t), \]

\[ \epsilon(t) = \epsilon(t) + \Phi(t) \sigma(t), \]

where \( \sigma(t) = [\sigma_1(t), \sigma_2(t)]^T \) and \( \Phi(t) \) is the estimate of \( \Sigma \Delta \), with \( \Phi_j(t) = \Psi_j(t) \delta_j(t) - h(z) [\Psi_j \varphi_j(t)] \) and \( \delta_j(t) = h(z) [\varphi_j(t)] \), \( j = 1, 2 \).

From (139), we see that \( \text{sign} \{d_1^j\} = \text{sign} \{d_2^j\} = -1 \). Using the above derivations, from (58)-(59), the parameter update laws are designed as

\[ \Psi_j(t + 1) = \Psi_j(t) - \frac{\gamma_j \epsilon_j \delta_j}{m^2(t)}, \]

\[ \Phi(t + 1) = \Phi(t) - \frac{\beta(t) \sigma(t)}{m^2(t)}, \]

where \( \beta, \gamma, j = 1, 2 \), are all chosen as 0.5, and \( m(t) = \sqrt{1 + \sum_{j=1}^{2} \sigma_j^2(t) + \sum_{j=1}^{2} \delta_j^2(t)} \delta_j(t). \)

### G. Simulation Results

We present two cases to verify the validity of the proposed adaptive control scheme: (I) the system output tracks a constant reference output signal \( y_1^* = [-1, 2]^T \); and (II) the system output tracks a time-varying reference output signal \( y_2^*(t) = [-1 - 0.5 \sin \frac{t}{2} + 0.5 \cos \frac{t}{2}, 2 + 0.5 \cos \frac{t}{2}]^T \).

For the case (I) study, Fig. 1 shows the response of the output signal \( y(t) \) of the model (135) versus the constant reference output signal \( y_1^* \), from which we see that the system output \( y(t) \) of the model (135) could track \( y_1^* \) asymptotically. Fig. 2 shows the response of the control signal \( u(t) \) and the system state variables, from which we see that the control input and state variables all converge to some bounded constant values. In addition, we also present the response of some parameter adaptation in Fig. 3, from which we see that \( \theta_i(t + 1) - \theta_i(t) \) converge to zero asymptotically (due to space limit, only four signals are given).

For the case (II) study, the operating procedure is similar to that for the case (I) study. Fig. 4 shows the response of the output signal of the model (135) versus the time-varying reference output signal \( y_2^*(t) \), from which we see that \( y(t) \) could track \( y_2^* \) asymptotically. Fig. 5 shows the response of the control signal \( u(t) \) and the system state variables, from which we see that the control input and state variables are all bounded. Fig. 6 presents the response of some parameter adaptation, from which we see that \( \theta_i(t + 1) - \theta_i(t) \) converge to zero asymptotically.
From the simulation results, we conclude that the proposed adaptive control scheme is valid.

VII. CONCLUDING REMARKS

In this paper, we have developed a new matrix decomposition based solution to the singularity problem in adaptive control of a class of non-canonical form MIMO DT nonlinear systems with vector relative degree \([1,1,\ldots,1]\). The developed control design overcomes the restrictive conditions that require the controlled plant in some canonical form and the positive/negative definiteness of the high-frequency gain matrix, and guarantees closed-loop stability and asymptotic output tracking. In addition, we also developed an implicit function based framework for nominal state feedback output tracking control for such systems with vector relative degree \([2,2,\ldots,2]\). Simulation results have demonstrated the control design procedure and verified the effectiveness of the proposed control scheme. Further work is needed to investigate the adaptive control details for such systems with high-order vector relative degrees, based on the nominal control scheme proposed in this paper.

REFERENCES


